Chapter 10

Addition of Angular Momentum

In this Chapter we study the problem of adding angular momenta. The way quantum angular momenta are added is more complex than what you are used to in the classical world, and also leads to several counterintuitive phenomena. However, in order to be able to add angular momenta properly, we first introduce the mathematical framework of a tensor product.

10.1 Mathematical Description of Systems with Many Degrees of Freedom

We will now describe the mathematical tools we will need to deal with the states and operators of two or more particles.

10.1.1 Tensor Product of States

Suppose we have two particles, labeled A and B. From the postulates of quantum mechanics we have enunciated at the beginning of this course, we know the state of the system comprising both particles, let's call it AB, must be described by a ray in a complex vector space, or in a complex Hilbert space, if A and B have continuous degrees of freedom. The natural question to ask is then, in what space does a generic state for the two particles, $|\psi_{AB}\rangle$, live in? If we call $\mathcal{H}_A$ and $\mathcal{H}_B$ the vector (Hilbert) spaces in which the quantum states of the individual particles live, then it is a postulate of quantum mechanics that a generic state vector describing the combined system lives in a space

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{11.1.1}$$

The symbol $\otimes$ refers to a *tensor product*, a mathematical operation that combines two vector (Hilbert) spaces to produce another one. The meaning of the tensor product is more easily understood in terms of explicit basis vectors, in the case of discrete vector spaces. For this purpose, let us assume that $\mathcal{H}_A$ is spanned by a set of basis vectors $\{|\mu_1\rangle, |\mu_2\rangle, |\mu_3\rangle, \dots |\mu_{n_A}\rangle\}$ and that $\mathcal{H}_B$ is spanned by a set of other basis vectors $\{|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle, \dots |\nu_{n_B}\rangle\}$. Then, the vector space $\mathcal{H}_{AB}$ is by construction spanned by basis vectors consisting of all the pairwise combinations of the basis vectors of A and B, and the basis states of the composite system are written as

$$|\mu_i\rangle \otimes |\nu_j\rangle \quad \forall\, i \in [1, n_A], j \in [1, n_B]. \tag{11.1.2}$$

The symbol $\otimes$ is a mathematical operation known as the *tensor product* or *outer product* of two vectors, that we are going to characterize more in detail in the following. For the moment, we can already see that the total number of basis states for the composite system is $n_A \times n_B$ and all quantum states in $\mathcal{H}_{AB}$ can be written as linear combinations of the composite basis states:

$$|\psi_{AB}\rangle = \sum_{ij} c_{ij} |\mu_i\rangle \otimes |\nu_j\rangle, \tag{11.1.3}$$

$$= \sum_{ij} c_{ij} |\lambda_{ij}\rangle. \tag{11.1.4}$$

with c_{ij} some complex coefficients, and where we have defined the basis vectors $|\lambda_{ij}\rangle \equiv |\mu_i\rangle \otimes |\nu_j\rangle$. In order to work with these states, we need to know how to perform inner products between states belonging to the tensor product space $\mathcal{H}_{AB}$. The inner product between two basis states is defined as

$$\langle \lambda_{ij} | \lambda_{kl} \rangle = \Big(\langle \mu_i | \otimes \langle \nu_j |\Big)\Big(|\mu_k\rangle \otimes |\nu_l\rangle\Big) \tag{11.1.5}$$

$$\equiv \langle \mu_i | \mu_k \rangle \langle \nu_j | \nu_l \rangle \tag{11.1.6}$$

$$= \delta_{ik}\delta_{jl}. \tag{11.1.7}$$

This definition is relatively easy to understand: the inner product is obtained as the product of the elementary (A or B) inner products. Also, it shows that the basis states of the composite system are orthogonal by construction. As a consequence, the inner product between two generic states of the composite system

$$|\phi\rangle = \sum_{ij} b_{ij} |\lambda_{ij}\rangle, \tag{11.1.8}$$

$$|\psi\rangle = \sum_{ij} c_{ij} |\lambda_{ij}\rangle, \tag{11.1.9}$$

reads

$$\langle \phi | \psi \rangle = \sum_{ij} \sum_{kl} b^*_{ij} c_{kl} \langle \lambda_{ij} | \lambda_{kl} \rangle \tag{11.1.10}$$

$$= \sum_{ij} \sum_{kl} b^*_{ij} c_{kl} \delta_{ik} \delta_{jl} \tag{11.1.11}$$

$$= \sum_{ij} b^*_{ij} c_{ij}. \tag{11.1.12}$$

We also see that the basis states of the composite system satisfy the closure relation:

$$\sum_{ij} |\lambda_{ij}\rangle\langle \lambda_{ij}| = \mathbb{I}. \tag{11.1.13}$$

Formally speaking, the tensor product satisfies all the intuitive properties you might expect from a product, for example, given a scalar a and two arbitrary vectors $|v\rangle \in \mathcal{H}_A$ and $|w\rangle \in \mathcal{H}_B$, we have

$$a(|v\rangle \otimes |w\rangle) = (a|v\rangle) \otimes |w\rangle = |v\rangle \otimes (a|w\rangle), \tag{11.1.14}$$

also, it is distributive, thus

$$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle, \tag{11.1.15}$$
$$|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle. \tag{11.1.16}$$

Finally, the construction of the product state space can be generalized from the case of two particles to the case of many particles, $A, B, C, \ldots$, since the composite vector (Hilbert) space will be simply given by the tensor product of the individual state spaces

$$\mathcal{H}_{ABC\ldots} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \ldots, \tag{11.1.17}$$

and in general, the resulting space will have a large dimension when we have many particles, since it is the product of the size of the individual dimensions

$$n_{ABC\ldots} = n_A \times n_B \times n_C \times \cdots . \tag{11.1.18}$$

Let us see an example of this formalism in the case of two spins 1/2, thus when $\mathcal{H}_A$ and $\mathcal{H}_B$ are both vector spaces of dimension 2. As basis states of the individual spins we take the eigenkets of S_z, thus the resulting tensor product space is given by the 4 states

$$|1\rangle = |+\rangle_A \otimes |+\rangle_B, \tag{11.1.19}$$
$$|2\rangle = |+\rangle_A \otimes |-\rangle_B, \tag{11.1.20}$$
$$|3\rangle = |-\rangle_A \otimes |+\rangle_B, \tag{11.1.21}$$
$$|4\rangle = |-\rangle_A \otimes |-\rangle_B, \tag{11.1.22}$$

and a generic state of two spins is written as

$$|\psi\rangle = \sum_{k=1}^{4} c_k |k\rangle, \tag{11.1.23}$$

where, as always, by definition

$$c_k = \langle k|\psi\rangle. \tag{11.1.24}$$

For example, take

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle_A \otimes |-\rangle_B - |-\rangle_A \otimes |+\rangle_B\Big), \tag{11.1.25}$$
$$= \frac{1}{\sqrt{2}}\Big(|2\rangle - |3\rangle\Big). \tag{11.1.26}$$

We can easily check that this is a physically valid state, since it is correctly normalized:

$$\langle\psi|\psi\rangle = \frac{1}{2}\Big(\langle 2|2\rangle + \langle 3|3\rangle\Big), \tag{11.1.27}$$
$$= 1. \tag{11.1.28}$$

10.1.2 Tensor Product of Operators

So far we have introduced the state space for a system of many particles but we haven't talked about the operators that act on this space, and how they are related to the measurement process. If we have two operators $\hat{T}_A$ and $\hat{T}_B$ acting on the individual spaces, the resulting operator that acts on the product space is also written as a tensor product:

$$\hat{T}_{AB} = \hat{T}_A \otimes \hat{T}_B, \tag{11.2.1}$$

where the resulting operator $\hat{T}_{AB}$ now acts on vectors in the space $\mathcal{H}_A \otimes \mathcal{H}_B$. The composite operator acts as follows:

$$\hat{T}_{AB}|\lambda_{ij}\rangle = \left(\hat{T}_A \otimes \hat{T}_B\right)\left(|\mu_i\rangle \otimes |\nu_j\rangle\right) \tag{11.2.2}$$

$$\equiv \left(\hat{T}_A|\mu_i\rangle\right) \otimes \left(\hat{T}_B|\nu_j\rangle\right), \tag{11.2.3}$$

thus, quite naturally, each of the two operators in the product act on the kets that belong to the respective vector spaces. As a special case, notice that if we are given only an operator that acts on one of the two subsystems, this is to be understood as

$$\hat{T}_{AB} = \hat{T}_A \otimes \hat{I}_B \tag{11.2.4}$$

if only $\hat{T}_A$ is given, and where $\hat{I}_B$ is the identity operator for subsystem B. Similarly,

$$\hat{T}'_{AB} = \hat{I}_A \otimes \hat{T}_B, \tag{11.2.5}$$

if only $\hat{T}_B$ is given. As a result, it is easy to see that these two operators, acting non-trivially only on one of the two subsystems, commute since:

$$\hat{T}'_{AB}\hat{T}_{AB}|\lambda_{ij}\rangle = \left(\hat{I}_A \otimes \hat{T}_B\right)\left(\hat{T}_A \otimes \hat{I}_B\right)\left(|\mu_i\rangle \otimes |\nu_j\rangle\right) \tag{11.2.6}$$

$$= \left(\hat{I}_A \otimes \hat{T}_B\right)\left(\hat{T}_A|\mu_i\rangle \otimes \hat{I}_B|\nu_j\rangle\right) \tag{11.2.7}$$

$$= \hat{T}_A|\mu_i\rangle \otimes \hat{T}_B|\nu_j\rangle, \tag{11.2.8}$$

$$\hat{T}_{AB}\hat{T}'_{AB}|\lambda_{ij}\rangle = \left(\hat{T}_A \otimes \hat{I}_B\right)\left(\hat{I}_A \otimes \hat{T}_B\right)\left(|\mu_i\rangle \otimes |\nu_j\rangle\right) \tag{11.2.9}$$

$$= \left(\hat{T}_A|\mu_i\rangle \otimes \hat{I}_B|\nu_j\rangle\right)\left(\hat{I}_A \otimes \hat{T}_B\right) \tag{11.2.10}$$

$$= \hat{T}_A|\mu_i\rangle \otimes \hat{T}_B|\nu_j\rangle. \tag{11.2.11}$$

thus

$$\left[\hat{T}_A \otimes \hat{I}_B, \hat{I}_A \otimes \hat{T}_B\right] = 0. \tag{11.2.12}$$

Let us give a concrete example for two spins 1/2, and imagine that we are interested in studying the *total* z component of the spin. If we call $\hat{S}_z^{(A)}$ and $\hat{S}_z^{(B)}$ the spin operators for the individual spins, such that

$$\hat{S}_z^{(A)}|m\rangle_A = \hbar m|m\rangle_A, \tag{11.2.13}$$

$$\hat{S}_z^{(B)}|m'\rangle_B = \hbar m'|m'\rangle_B, \tag{11.2.14}$$

for $m, m' = \pm 1/2$, then it is natural to define the total spin as the sum of these two operators. In order to do so, however, we need to recall that these operators are acting on different spaces, thus before summing them up we need to "upgrade" them to be good operators for the composite vector space. Thus the total $\hat{S}_z^{(AB)}$ operator reads:

$$\hat{S}_z^{(AB)} = \hat{S}_z^{(A)} \otimes \hat{I}^{(B)} + \hat{I}^{(A)} \otimes \hat{S}_z^{(B)}. \tag{11.2.15}$$

It is then straightforward to see how this operator acts on a general state. For example, if we take a basis vector for the composite system, we have

$$\hat{S}_z^{(AB)}\Big(|m\rangle_A \otimes |m'\rangle_B\Big) = \Big(\hat{S}_z^{(A)} \otimes \hat{I}^{(B)} + \hat{I}^{(A)} \otimes \hat{S}_z^{(B)}\Big)\Big(|m\rangle_A \otimes |m'\rangle_B\Big) \tag{11.2.16}$$

$$= \Big(\hat{S}_z^{(A)}|m\rangle_A\Big) \otimes |m'\rangle_B + |m\rangle_A \otimes \Big(\hat{S}_z^{(B)}|m'\rangle_B\Big) \tag{11.2.17}$$

$$= \hbar m\Big(|m\rangle_A \otimes |m'\rangle_B\Big) + \hbar m'\Big(|m\rangle_A \otimes |m'\rangle_B\Big) \tag{11.2.18}$$

$$= \hbar(m + m')\Big(|m\rangle_A \otimes |m'\rangle_B\Big), \tag{11.2.19}$$

thus the composite state is an eigen-ket of the total spin, with an eigenvalue $\hbar(m + m')$ that is the sum of the individual eigenvalues.

10.2 Addition of Generic Angular Momenta

Given two particles with given angular momenta operators, say $\hat{\mathbf{J}}_{(1)}$ and $\hat{\mathbf{J}}_{(2)}$, we would like to study the total angular momentum that these two particles have. The resulting angular momentum is the sum of the individual momenta, however we have to be careful when performing the sum and recall that the two operators act on distinct Hilbert spaces, $\mathcal{H}_{(1)}$ and $\mathcal{H}_{(2)}$. As discussed in the previous section, the correct way of summing the two operators is then to first "upgrade" them to act on the same Hilbert space $\mathcal{H} = \mathcal{H}_{(1)} \otimes \mathcal{H}_{(2)}$, and then consider the sum. The total angular momentum operator is then to be defined as

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_{(1)} \otimes \hat{I}_{(2)} + \hat{I}_{(1)} \otimes \hat{\mathbf{J}}_{(2)} \tag{12.1.1}$$

$$= \Big(\hat{J}_{(1)x} \otimes \hat{I}_{(2)} + \hat{I}_{(1)} \otimes \hat{J}_{(2)x}, \ldots\Big) \tag{12.1.2}$$

$$= \Big(\hat{J}_x, \hat{J}_y, \hat{J}_z\Big). \tag{12.1.3}$$

Moreover, the two angular momentum operators $\hat{\mathbf{J}}_{(1)}$ and $\hat{\mathbf{J}}_{(2)}$ act on different spaces, thus their components commute:

$$\Big[\hat{J}_{(1)\alpha} \otimes \hat{I}_{(2)}, \hat{I}_{(1)} \otimes \hat{J}_{(2)\beta}\Big] = 0. \tag{12.1.4}$$

This is easily shown using the tensor product notation:

$$\left(\hat{J}_{(1)\alpha} \otimes \hat{I}_{(2)}\right)\left(\hat{I}_{(1)} \otimes \hat{J}_{(2)\beta}\right)|\Psi\rangle_1 \otimes |\Psi\rangle_2 = \left(\hat{I}_{(1)} \otimes \hat{J}_{(2)\beta}\right)\left(\hat{J}_{(1)\alpha}|\Psi\rangle_1\right) \otimes |\Psi\rangle_2 \tag{12.1.5}$$

$$= \left(\hat{J}_{(1)\alpha}|\Psi\rangle_1\right) \otimes \left(\hat{J}_{(2)\beta}|\Psi\rangle_2\right). \tag{12.1.6}$$

and similarly for the second term of the commutator, to find that indeed the latter vanishes. Since it is quite cumbersome to carry around the tensor product symbols, in the following we will use a slightly *wrong* but widely adopted notation, in which we write the total angular momentum operator as

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_{(1)} + \hat{\mathbf{J}}_{(2)}. \tag{12.1.7}$$

This notation is compact but possibly also dangerous, because you might be tempted to assume (wrongly) that $\hat{\mathbf{J}}_{(1)}$ and $\hat{\mathbf{J}}_{(2)}$ act on the same Hilbert space. However, we have stressed many times now that this is not the case. So, just be careful when using this notation, and if in doubt, just go back to the tensor product one! In compact notation, the commutation relations for the components of the angular momenta read

$$[\hat{J}_{(1)\alpha}, \hat{J}_{(2)\beta}] \;=\; 0\,, \tag{10.2.1}$$

As a consequence of this relationship, we can also easily verify that the total angular momentum operator is still a valid angular momentum operator, in the sense that it satisfies the usual commutation relations. We can check this explicitly:

$$[\hat{J}_\alpha, \hat{J}_\beta] = [\hat{J}_{(1)\alpha} + \hat{J}_{(2)\alpha}, \hat{J}_{(1)\beta} + \hat{J}_{(2)\beta}] \tag{10.2.2}$$

$$= [\hat{J}_{(1)\alpha}, \hat{J}_{(1)\beta}] + [\hat{J}_{(2)\alpha}, \hat{J}_{(2)\beta}] \tag{10.2.3}$$

$$= i\hbar\, \epsilon_{\alpha,\beta,\gamma} \hat{J}_{(1)\gamma} + i\hbar\, \epsilon_{\alpha,\beta,\gamma} \hat{J}_{(2)\gamma} \tag{10.2.4}$$

$$= i\hbar\, \epsilon_{\alpha,\beta,\gamma} \hat{J}_\gamma\,. \tag{10.2.5}$$

Physically speaking, the total angular momentum operator then must be also associated to a rotation operator

$$\hat{D}(\boldsymbol{\theta}) \;=\; e^{-\frac{i}{\hbar}\hat{\boldsymbol{J}}\boldsymbol{\theta}}\,, \tag{10.2.6}$$

The meaning of this rotation operator is clarified considering the product of the two rotation operators acting on each of the two subsystems, namely

$$\hat{D}_{(1)}(\boldsymbol{\theta})\, \hat{D}_{(2)}(\boldsymbol{\theta}) = e^{-\frac{i}{\hbar}\hat{J}_{(1)}\boldsymbol{\theta}}\, e^{-\frac{i}{\hbar}\hat{J}_{(2)}\boldsymbol{\theta}} \tag{10.2.7}$$

$$= e^{-\frac{i}{\hbar}\hat{\boldsymbol{J}}\boldsymbol{\theta}}\,, \tag{10.2.8}$$

where in the last line we have used the fact that $[\hat{J}_{(1)\alpha}, \hat{J}_{(2)\beta}] = 0$, thus the product of the two exponentials can be absorbed into a single exponential of the sum. From this expression we also deduce that the rotation operator associated to the total angular momentum corresponds to taking rotations of the coordinate systems of both particles at the same time,

$$\hat{D}(\boldsymbol{\theta}) \;=\; \hat{D}_{(1)}(\boldsymbol{\theta}) \otimes \hat{D}_{(2)}(\boldsymbol{\theta})\,. \tag{10.2.9}$$

Moreover, since $\hat{J}$ is just another angular momentum operator, it will also have a set of eigenvalues and eigenvectors of the "standard" form:

$$\hat{J}^2 \left|j,m\right\rangle = \hbar^2 j(j+1) \left|j,m\right\rangle \tag{10.2.10}$$

$$\hat{J}_z \left|j,m\right\rangle = \hbar m \left|j,m\right\rangle \;. \tag{10.2.11}$$

This expression however does not tell the whole story, since there are other quantities that commute with $\hat{J}^2$ and $\hat{J}_z$. We can verify for example that the total momentum squared commutes with the individual total momenta squared. To prove this, we start writing

$$\begin{aligned}
\hat{J}^2 &= \left(\hat{J}_{(1)x} + \hat{J}_{(2)x}\right)^2 + \left(\hat{J}_{(1)y} + \hat{J}_{(2)y}\right)^2 + \left(\hat{J}_{(1)z} + \hat{J}_{(2)z}\right)^2 \\
&= \hat{J}^2_{(1)} + \hat{J}^2_{(2)} + 2\hat{J}_{(1)x}\hat{J}_{(2)x} + 2\hat{J}_{(1)y}\hat{J}_{(2)y} + 2\hat{J}_{(1)z}\hat{J}_{(2)z} \\
&= \hat{J}^2_{(1)} + \hat{J}^2_{(2)} + 2\hat{J}_{(1)z}\hat{J}_{(2)z} + \tfrac{1}{2}\left(\hat{J}^+_{(1)} + \hat{J}^-_{(1)}\right)\left(\hat{J}^+_{(2)} + \hat{J}^-_{(2)}\right) \\
&\quad - \tfrac{1}{2}\left(\hat{J}^+_{(1)} - \hat{J}^-_{(1)}\right)\left(\hat{J}^+_{(2)} - \hat{J}^-_{(2)}\right) \\
&= \hat{J}^2_{(1)} + \hat{J}^2_{(2)} + 2\hat{J}_{(1)z}\hat{J}_{(2)z} + \hat{J}^+_{(1)}\hat{J}^-_{(2)} + \hat{J}^-_{(1)}\hat{J}^+_{(2)}\,,
\end{aligned}$$

thus since $[\hat{J}^2_{(1)}, \hat{J}^{\pm}_{(1)}] = [\hat{J}^2_{(1)}, \hat{J}^z_{(1)}] = 0$, and similarly for the particle 2, we have

$$[\hat{J}^2, \hat{J}^2_{(1)}] = 0, \tag{10.2.12}$$

$$[\hat{J}^2, \hat{J}^2_{(2)}] = 0. \tag{10.2.13}$$

Moreover, two individual squared momenta also commute with the total $\hat{J}_z$, since

$$[\hat{J}_z, \hat{J}^2_{(1)}] = [\hat{J}_{(1)z} + \hat{J}_{(2)z}, \hat{J}^2_{(1)}] \tag{10.2.14}$$

$$= [\hat{J}_{(1)z}, \hat{J}^2_{(1)}] \tag{10.2.15}$$

$$= 0, \tag{10.2.16}$$

$$[\hat{J}_z, \hat{J}^2_{(2)}] = [\hat{J}_{(1)z} + \hat{J}_{(2)z}, \hat{J}^2_{(2)}] \tag{10.2.17}$$

$$= [\hat{J}_{(2)z}, \hat{J}^2_{(2)}] \tag{10.2.18}$$

$$= 0. \tag{10.2.19}$$

This means that we have four mutually commuting quantities, $\hat{J}^2$, $\hat{J}^2_{(1)}$, $\hat{J}^2_{(2)}$ and $\hat{J}_z$ whose eigenvalues can be used to index the eigen-kets of the total momentum such that

$$\hat{J}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j+1) |j_1, j_2; j, m\rangle , \tag{10.2.20}$$

$$\hat{J}_z |j_1, j_2; j, m\rangle = \hbar m |j_1, j_2; j, m\rangle , \tag{10.2.21}$$

$$\hat{J}^2_{(1)} |j_1, j_2; j, m\rangle = \hbar^2 j_1(j_1+1) |j_1, j_2; j, m\rangle , \tag{10.2.22}$$

$$\hat{J}^2_{(2)} |j_1, j_2; j, m\rangle = \hbar^2 j_2(j_2+1) |j_1, j_2; j, m\rangle . \tag{10.2.23}$$

10.2.1 Tensor-Product Basis

While the representation we have introduced above is the "standard" representation of the composed momenta, it is not very convenient to work with. It is in fact more natural to introduce basis states that are simultaneous eigen-kets of the individual components, and that we have already analyzed in the previous Chapters. We therefore consider the tensor-product basis states

$$|j_1, j_2; m_1, m_2\rangle \;=\; |j_1, m_1\rangle \otimes |j_2, m_2\rangle , \tag{10.2.24}$$

that are simultaneous eigen-kets of $\hat{J}^2_{(1)}, \hat{J}^2_{(2)}, \hat{J}_{(1)z}, \hat{J}_{(2)z}$. These four operators are obviously mutually commuting, since operators with different particle indexes act on different Hilbert spaces (thus commute) and same-particle operators commute as well, i.e. we already know that $[\hat{J}_{(1)}, \hat{J}_{(1)z}] = [\hat{J}_{(2)}, \hat{J}_{(2)z}] = 0$. The basis eigen-kets we consider then satisfy

$$\hat{J}^2_{(1)} |j_1, j_2; m_1, m_2\rangle = \hbar^2 j_1(j_1+1) |j_1, j_2; m_1, m_2\rangle , \tag{10.2.25}$$

$$\hat{J}^2_{(2)} |j_1, j_2; m_1, m_2\rangle = \hbar^2 j_2(j_2+1) |j_1, j_2; m_1, m_2\rangle , \tag{10.2.26}$$

$$\hat{J}_{(1)z} |j_1, j_2; m_1, m_2\rangle = \hbar m_1 |j_1, j_2; m_1, m_2\rangle , \tag{10.2.27}$$

$$\hat{J}_{(2)z} |j_1, j_2; m_1, m_2\rangle = \hbar m_2 |j_1, j_2; m_1, m_2\rangle . \tag{10.2.28}$$

While this basis is convenient, the tensor-product states however are not eigenstates of the total momentum squared. This is because we cannot diagonalize at the same time the four operators above ($\hat{J}^2_{(1)}, \hat{J}^2_{(2)}, \hat{J}_{(1)z}, \hat{J}_{(2)z}$) and also $\hat{J}^2$. This can be checked noticing that for example $\hat{J}^2$ does not commute with the single-particle $\hat{J}_z$ operators:

$$[\hat{J}_{(1)z}, \hat{J}^2] = [\hat{J}_{(1)z}, \hat{J}^2_{(1)} + \hat{J}^2_{(2)} + 2\hat{J}_{(1)x}\hat{J}_{(2)x} + \hat{J}^+_{(1)}\hat{J}^-_{(2)} + \hat{J}^-_{(1)}\hat{J}^+_{(2)}] \tag{10.2.29}$$

$$= [\hat{J}_{(1)z}, \hat{J}^+_{(1)}\hat{J}^-_{(2)} + \hat{J}^-_{(1)}\hat{J}^+_{(2)}] \tag{10.2.30}$$

$$= \hat{J}_{(2)z}[\hat{J}_{(1)z}, \hat{J}^+_{(1)}] + [\hat{J}_{(1)z}, \hat{J}^-_{(1)}]\hat{J}^+_{(2)} \tag{10.2.31}$$

$$= \hbar\hat{J}_{(2)z}\hat{J}^+_{(1)} - \hbar\hat{J}^-_{(1)}\hat{J}^+_{(2)} \tag{10.2.32}$$

$$\neq 0. \tag{10.2.33}$$

Nonetheless, we can still use this convenient basis to express the eigen-kets of the total angular momentum squared, i.e. we can develop the eigen-kets as

$$|j_1, j_2; j, m\rangle \;=\; \sum_{m_1 m_2} |j_1, j_2; m_1, m_2\rangle \, \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle \,. \tag{10.2.34}$$

The coefficients of this transformation,

$$C^{jm}_{j_1 m_1 j_2 m_2} \;\equiv\; \Big\langle j_1, j_2; m_1, m_2 \;\Big|\; j_1, j_2; j, m \Big\rangle, \tag{10.2.35}$$

are called *Clebsch–Gordan coefficients* and play the vital role of connecting the two representations.

10.2.2 Properties of Clebsch–Gordan coefficients

There is a number of important properties of these coefficients that we can already deduce at this stage. First of all, the coefficients vanish unless

$$m \;=\; m_1 + m_2. \tag{10.2.36}$$

This can be proven noticing that

$$\left(\hat{J}_z - \hat{J}_{(1)z} - \hat{J}_{(2)z}\right) |j_1, j_2; j, m\rangle \;=\; 0, \tag{10.2.37}$$

thus multiplying this equation by $\langle j_1, j_2; m_1, m_2|$ we have

$$(m - m_1 - m_2)\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle \;=\; 0, \tag{10.2.38}$$

implying that the Clebsch–Gordan coefficients (appearing in the left-hand side of this equation) must vanish unless $m = m_1 + m_2$. This condition is quite natural, since it tells us that the total $\hat{J}_z$ has an eigenvalue which is the sum of the two individual eigenvalues of $\hat{J}_{(1)z}$ and $\hat{J}_{(2)z}$.
The other important condition is on the possible values that j can take; as it turns out that

$$|j_1 - j_2| \;\leq\; j \;\leq\; j_1 + j_2. \tag{10.2.39}$$

In order to see why this is the case, remember that

$$-j \;\leq\; m \;\leq\; j, \tag{10.2.40}$$

$$-j_1 \;\leq\; m_1 \;\leq\; j_1, \tag{10.2.41}$$

$$-j_2 \;\leq\; m_2 \;\leq\; j_2. \tag{10.2.42}$$

Now, if we set $m = j$ and $j_1 = m_1$, the inequality for m_2 (which is $m_2 = m - m_1$) becomes

$$-j_2 \;\leq\; j - j_1 \;\leq\; j_2, \tag{10.2.43}$$

$$j_1 - j_2 \;\leq\; j \;\leq\; j_1 + j_2. \tag{10.2.44}$$

Also, if we take $m = j$ and $j_2 = m_2$, the inequality for m_1 $(= m - m_2)$ becomes

$$-j_1 \;\leq\; j - j_2 \;\leq\; j_1, \tag{10.2.45}$$

$$j_2 - j_1 \;\leq\; j \;\leq\; j_1 + j_2, \tag{10.2.46}$$

thus we conclude with Eq. (10.2.39).

State Counting

An alternative way of convincing ourselves that Eq. (10.2.39) must be true is by counting the basis states in the two representations. In other words, we can count how many basis states $|j_1, j_2; m_1, m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$ exist and how many states $|j_1, j_2; j, m\rangle$ exist. In the first case, we know that the total number of states is given by the product of the number of states spanned by the individual kets that are taken into the tensor product, thus

$$N_I = \Big(2j_1 + 1\Big)\Big(2j_2 + 1\Big). \tag{10.2.47}$$

In the other case, we know that for each j there are $2j + 1$ states and if the inequality, Eq. (10.2.39), is satisfied we have that j must run between $(j_1 - j_2)$ and $(j_1 + j_2)$, assuming, as we can always do, that we pick $j_1 \geq j_2$. This means that the total number of states in the second count is (the full summation of the series is left as an exercise):

$$N_{II} = \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) \tag{10.2.48}$$

$$= (2j_1 + 1)(2j_2 + 1), \tag{10.2.49}$$

thus we find the consistent result $N_I = N_{II}$.

10.2.3 Explicit form of the Clebsch–Gordan coefficients

Determining explicit and general expressions for Clebsch–Gordan is a time–consuming and not very productive exercise that is still reason of nightmares for generations of students who were forced to derive them. We just quote here the final result, so that you can understand the reason of such nightmares:

$$\begin{aligned} C^{jm}_{j_1 m_1 j_2 m_2} &= \delta_{m_1+m_2,\,m} \sqrt{\frac{2j+1}{(j_1+j_2-j)!\,(j_1-j_2+j)!\,(j_2-j_1+j)!\,(j_1+j_2+j+1)!}} \\ &\times \sqrt{(j+m)!\,(j-m)!\,(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!} \\ &\times \sum_k (-1)^k \begin{pmatrix} j_1+j_2-j \\ k \end{pmatrix} \begin{pmatrix} j_1-j_2+j \\ j_1-m_1-k \end{pmatrix} \begin{pmatrix} j_2-j_1+j \\ j_2+m_2-k \end{pmatrix}. \end{aligned} \tag{10.2.50}$$

We also recall here a few more properties of the Clebsch–Gordan coefficients. They are real–valued by convention, and they satisfy the closure conditions

$$\sum_{jm} C^{jm}_{j_1 m_1 j_2 m_2}\, C^{jm}_{j_1 m_1' j_2 m_2'} = \delta_{m_1, m_1'}\, \delta_{m_2, m_2'}, \tag{10.2.51}$$

$$\sum_{m_1 m_2} C^{jm}_{j_1 m_1 j_2 m_2}\, C^{j'm'}_{j_1 m_1 j_2 m_2} = \delta_{j,j'}\, \delta_{m,m'}. \tag{10.2.52}$$

10.3 Example: Two spin-1/2 particles

We first consider a simple example of the formalism developed so far, where we can easily find an explicit representation for $|j_1, j_2; j, m\rangle$ by bypassing the explicit calculation of the Clebsch–Gordan coefficients. The example consists in forming the total angular momentum resulting from two spins 1/2. Formally, we form the vector operator

$$\hat{\mathbf{S}} = \hat{\mathbf{S}}_{(1)} + \hat{\mathbf{S}}_{(2)}, \tag{10.3.1}$$

$$= (\hat{S}_x, \hat{S}_y, \hat{S}_z). \tag{10.3.2}$$

The "convenient" basis in this case is then simply

$$|s_1, s_2; m_1, m_2\rangle = |s_1, m_1\rangle \otimes |s_2, m_2\rangle, \tag{10.3.3}$$

$$|m_1, m_2\rangle = |\pm, \pm\rangle, \tag{10.3.4}$$

where in the last line we omitted the $s_1 = s_2 = 1/2$ quantum numbers and just concentrated on the two possible values of $m_1 = \pm 1/2$ and $m_2 = \pm 1/2$. In total then we have four states

$$|1\rangle = |++\rangle, \quad |2\rangle = |+-\rangle, \quad |3\rangle = |-+\rangle, \quad |4\rangle = |--\rangle.$$

In order to find the states $|j_1, j_2; j, m\rangle$ in the "standard" representation, we start by explicitly computing the total spin squared

$$\hat{S}^2 = \hat{S}^2_{(1)} + \hat{S}^2_{(2)} + 2\hat{S}_{(1)z}\hat{S}_{(2)z} + \hat{S}^+_{(1)}\hat{S}^-_{(2)} + \hat{S}^-_{(1)}\hat{S}^+_{(2)} \tag{10.3.5}$$

$$= \frac{3}{4}\hbar^2\Big(\hat{I}_{(1)} + \hat{I}_{(2)}\Big) + 2\hat{S}_{(1)z}\hat{S}_{(2)z} + \hat{S}^+_{(1)}\hat{S}^-_{(2)} + \hat{S}^-_{(1)}\hat{S}^+_{(2)}. \tag{10.3.6}$$

From this expression we see that the last two terms are vanishing when applied to the states $|{+}{+}\rangle$ and $|{-}{-}\rangle$ since, for example,

$$\hat{S}^+_{(1)}|{+}{+}\rangle = \hat{S}^+_{(1)} \otimes \hat{I}_{(2)}\Big(|+\rangle_1 \otimes |+\rangle_2\Big) \tag{10.3.7}$$

$$= \Big(\hat{S}^+_{(1)}|+\rangle_1\Big) \otimes |+\rangle_2 \tag{10.3.8}$$

$$= 0. \tag{10.3.9}$$

Furthermore, we can easily verify that $|{+}{+}\rangle$ is an eigenstate of $\hat{S}^2$, since

$$\hat{S}^2|{+}{+}\rangle = \left(\tfrac{3}{4}\hbar^2 2 + 2\hbar^2\tfrac{1}{2}\tfrac{1}{2}\right)|{+}{+}\rangle \tag{10.3.10}$$

$$= 2\hbar^2|{+}{+}\rangle, \tag{10.3.11}$$

and likewise for $|{-}{-}\rangle$,

$$\hat{S}^2|{-}{-}\rangle = \left(\tfrac{3}{4}\hbar^2 2 + 2\hbar^2\Big(-\tfrac{1}{2}\Big)\Big(-\tfrac{1}{2}\Big)\right)|{-}{-}\rangle \tag{10.3.12}$$

$$= 2\hbar^2|{-}{-}\rangle. \tag{10.3.13}$$

With eigenvalue $2\hbar^2$. However, from the general theory we know that the eigenvalues of $\hat{S}^2$ are also equal to $\hbar^2 s(s+1)$; thus we conclude that these two states have $s = 1$.

We have then successfully found the first two states we were looking for in the standard representation:

$$|j=1;m=1\rangle = |{+}{+}\rangle\,, \tag{10.3.14}$$

$$|j=1;m=-1\rangle = |{-}{-}\rangle\,. \tag{10.3.15}$$

From the general theory of angular momentum, however, we know that the $j=1$ states always come as a triplet of states ($m=-1,0,1$); thus there must be still a missing state we have not found yet with $j=1, m=0$. In order to find it, we apply the lowering operator

$$\begin{aligned}\hat{S}^- &= \hat{S}_x - i\hat{S}_y &(10.3.16)\\ &= \hat{S}_{(1)x} + \hat{S}_{(2)x} - i\hat{S}_{(1)y} - i\hat{S}_{(2)y} &(10.3.17)\\ &= \hat{S}^-_{(1)} + \hat{S}^-_{(2)}. &(10.3.18)\end{aligned}$$

to the state with highest m:

$$\hat{S}^-\,|j=1;m=1\rangle = \hat{S}^-_{(1)}\,|{+}{+}\rangle + \hat{S}^-_{(2)}\,|{+}{+}\rangle \tag{10.3.19}$$

$$\begin{aligned}\hbar\sqrt{j(j+1)-m(m-1)}\,|j=1;m=0\rangle &= \hbar\sqrt{s_1(s_1+1)-m_1(m_1-1)}\,|{-}{+}\rangle\\ &\quad+\hbar\sqrt{s_2(s_2+1)-m_2(m_2-1)}\,|{+}{-}\rangle\,, &(10.3.20)\end{aligned}$$

$$\sqrt{2}\,|j=1;m=0\rangle = |{-}{+}\rangle + |{+}{-}\rangle\,. \tag{10.3.21}$$

From the last line we can read out the third state with $j=1, m=0$ we were missing before:

$$|j=1;m=0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|{-}{+}\rangle + |{+}{-}\rangle\Big). \tag{10.3.22}$$

To find the last and final state (remember that we started with four states for the "convenient" basis, so we need to find also four states in the standard basis) we realize that the missing state must be the one with $|j=0;m=0\rangle$ (that is the only allowed value of j remaining, from the inequality condition (10.2.39)). This state is found imposing that it is orthogonal to all the other states we have already found. We start by imposing that it is orthogonal to the other state we found for $m=0$:

$$\langle j=0,m=0|j=1;m=0\rangle \;=\; 0, \tag{10.3.23}$$

and we obtain:

$$|j=0;m=0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|-+\rangle - |+-\rangle\Big). \tag{10.3.24}$$

It can be easily checked that this state is also orthogonal to all the other states previously found, just because they carry different m; thus all products such as $\langle - - \,|+ +\rangle$ vanish. To summarize, we have found the four states we were looking for in the "standard" representation:

$$|j=1;m=1\rangle = |++\rangle\,, \tag{10.3.25}$$

$$|j=1;m=0\rangle = \frac{1}{\sqrt{2}}\Big(|-+\rangle + |+-\rangle\Big), \tag{10.3.26}$$

$$|j=1;m=-1\rangle = |--\rangle\,, \tag{10.3.27}$$

$$|j=0;m=0\rangle = \frac{1}{\sqrt{2}}\Big(|-+\rangle - |+-\rangle\Big). \tag{10.3.28}$$

with the first three having $j=1$ (the triplet) and the last one with $j=0$ (the singlet).

10.4 Adding Spin and Orbital Momentum

Another example we propose here is the important case of adding spin and orbital angular-momentum degrees of freedom. For example, we can form the total angular momentum of a particle with spin:

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}, \tag{10.4.1}$$

$$= \hat{L}\otimes\hat{I}_S + \hat{I}_L\otimes\hat{S}, \tag{10.4.2}$$

where the second line once more emphasizes that the two operators act on different Hilbert spaces. Notice that actually in this case the first operator (the orbital angular momentum) acts on an infinite Hilbert space, whereas the spin operator acts on a finite vector space. As before, we can interpret the resulting rotation operator as the product of two independent rotations on the respective degrees of freedom:

$$\hat{D}(\boldsymbol{\theta}) = \exp\!\Big(-\frac{i}{\hbar}\,\hat{\mathbf{L}}\cdot\boldsymbol{\theta}\Big)\otimes\exp\!\Big(-\frac{i}{\hbar}\,\hat{\mathbf{S}}\cdot\boldsymbol{\theta}\Big). \tag{10.4.3}$$

A typical way of writing the wave function of a particle with spin (say, an electron) is by means of the tensor-product basis $|r\rangle\otimes|s,m\rangle$ such that the state vector is

$$\Big(\langle r|\otimes\langle s,m|\Big)\,|\Psi\rangle \;=\; \Psi(r,m), \tag{10.4.4}$$

where the first variable $r=(x,y,z)$ is continuous, whereas the second one $m=\pm\frac{1}{2}$ is discrete. An alternative way to write the state is as a vector of two continuous-space wave functions:

$$\begin{pmatrix}\Psi_+(r)\\ \Psi_-(r)\end{pmatrix}, \tag{10.4.5}$$

so that $\Psi_\pm(r)=\Psi(r,\pm\frac{1}{2})$. For example $|\Psi_+(r)|^2$ gives the probability density of finding a spin up at position r. This representation is also called spin–orbital.

Standard basis

The spin-orbital representation is typically enough for most applications, but we might also want the "standard" representation involving the eigenvalues of $\hat{J}^2, \hat{J}_z, \hat{S}^2, \hat{L}^2$. We call this basis

$$|l, s; j, m\rangle \,, \tag{10.4.6}$$

whereas the "convenient" tensor-product basis is

$$|l, m_l, s, m_s\rangle \;=\; |l, m_l\rangle \otimes |s, m_s\rangle \,. \tag{10.4.7}$$

In what follows we omit the explicit value of $s = \frac{1}{2}$ and l from all kets. From Eq. (10.2.39) there are only two allowed values of j, namely $j_{\text{max}} = l + \frac{1}{2}$ and $j_{\text{min}} = l - \frac{1}{2}$. As for two spins, the state with $j_{\text{max}} = m_{\text{max}} = l + \frac{1}{2}$ is

$$\left|j = l + \tfrac{1}{2},\, m = l + \tfrac{1}{2}\right\rangle = |l, l\rangle \otimes |+\rangle \,. \tag{10.4.8}$$

Applying the total spin lowering operator $\hat{J}^- = \hat{L}^- + \hat{S}^-$ and using $\hat{J}^- |j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\, |j, m-1\rangle$ we get

$$\hat{J}^- \left|j = l + \tfrac{1}{2},\, m = l + \tfrac{1}{2}\right\rangle = \hbar\sqrt{2l+1}\,\left|j = l + \tfrac{1}{2},\, m = l - \tfrac{1}{2}\right\rangle \tag{10.4.9}$$

$$= \hat{L}^- |l, l\rangle \otimes |+\rangle + |l, l\rangle \otimes \hat{S}^- |+\rangle \tag{10.4.10}$$

$$= \hbar\Big(\sqrt{2l}\, |l, l-1\rangle \otimes |+\rangle + |l, l\rangle \otimes |-\rangle\Big). \tag{10.4.11}$$

thus

$$\left|j = l + \tfrac{1}{2},\, m = l - \tfrac{1}{2}\right\rangle \;=\; \sqrt{\frac{2l}{2l+1}}\; |l, l-1\rangle \otimes |+\rangle + \sqrt{\frac{1}{2l+1}}\; |l, l\rangle \otimes |-\rangle \,. \tag{10.4.12}$$

Iterating the procedure gives the general relation

$$\left|j = l + \tfrac{1}{2},\, m\right\rangle \;=\; \sqrt{\frac{l + m + \frac{1}{2}}{2l+1}}\; \left|l,\, m - \tfrac{1}{2}\right\rangle \otimes |+\rangle + \sqrt{\frac{l - m + \frac{1}{2}}{2l+1}}\; \left|l,\, m + \tfrac{1}{2}\right\rangle \otimes |-\rangle \,. \tag{10.4.13}$$

For the multiplet with $j = l - \frac{1}{2}$ the highest-m state is a linear combination

$$\left|j = l - \tfrac{1}{2},\, m = l - \tfrac{1}{2}\right\rangle \;=\; c_1\, |l,\, l-1\rangle \otimes |+\rangle + c_2\, |l,\, l\rangle \otimes |-\rangle \,, \tag{10.4.14}$$

whose coefficients follow from the orthogonality condition

$$\langle j = l + \tfrac{1}{2},\, m = l - \tfrac{1}{2} | j = l - \tfrac{1}{2},\, m = l - \tfrac{1}{2}\rangle = 0, \tag{10.4.15}$$

$$c_1\sqrt{\frac{2l}{2l+1}} + c_2\sqrt{\frac{1}{2l+1}} = 0. \tag{10.4.16}$$

Choosing $c_1 = \sqrt{1/(2l+1)}$ and $c_2 = -\sqrt{2l/(2l+1)}$ (using the normalization condition) gives

$$\left|j = l - \tfrac{1}{2},\, m = l - \tfrac{1}{2}\right\rangle \;=\; \sqrt{\frac{1}{2l+1}}\; |l,\, l-1\rangle \otimes |+\rangle - \sqrt{\frac{2l}{2l+1}}\; |l,\, l\rangle \otimes |-\rangle\,. \qquad (10.4.17)$$

Requiring orthogonality with the corresponding states in the $j = l + \frac{1}{2}$ multiplet,

$$\langle j = l + \tfrac{1}{2},\, m | j = l - \tfrac{1}{2},\, m\rangle = 0, \qquad (10.4.18)$$

finally yields

$$\left|j = l - \tfrac{1}{2},\, m\right\rangle \;=\; \sqrt{\frac{l - m + \frac{1}{2}}{2l+1}}\; \left|l,\, m - \tfrac{1}{2}\right\rangle \otimes |+\rangle - \sqrt{\frac{l + m + \frac{1}{2}}{2l+1}}\; \left|l,\, m + \tfrac{1}{2}\right\rangle \otimes |-\rangle\,.$$

10.5 References and Further Reading

The discussion in this Chapter is mainly adapted and simplified from Sakurai, Chapter 3 (Section 3.8). Cohen-Tannoudji's book discusses the addition of angular momentum in Volume 2 (Chapter 10). The complements to that Chapter (especially A_X and B_X) contain many details on Clebsch–Gordan coefficients and additional examples that can deepen one's technical understanding of the topic.